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Stability Criteria for Breached Pair Superfluidity

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We present simple, concrete, two-fermion models that exhibit thermodynamically stable isotropic translationally-invariant gapless superfluid states (breached pair superfluidity). The momentum structure of the pairing interaction and the mass ratio between the components are crucial for determining the stability of such states.

INTRODUCTION

Recently there has been interest in superfluid fermion systems where there exist superfluid states that retain gapless fermionic excitations [1, 2, 3, 4]. These states embody "phase separation in momentum space": some degrees of freedom pair, forming a superfluid, while others remain unpaired, maintaining normal state properties of a Fermi surface. They are likely to become experimentally accessible in the near future [5].

We shall consider systems of two species that, in the absence of interactions, would have two distinct Fermi surfaces. Simple heuristic considerations suggest the possibility that pairing takes place about the Fermi surfaces, but that there is no pairing in a region between the surfaces: this led to the term "breached pair" [2]. A breached pair superfluid state (BP) is characterized by the coexistence of a superfluid and a normal component in a translationally invariant and isotropic state. These components are accommodated in different regions of momentum space with the normal component residing in the "breaches", bounded by gapless Fermi surfaces.

A state of this type was considered much earlier by Sarma [6]. He considered the case of a superconductor in an external magnetic field, and found that, although there is a self-consistent mean-field solution with gapless modes, it is unfavored energetically to the fully gapped BCS solution. Similar results were considered in the context of color superconductivity [7], again concluding that these states are not stable at fixed chemical potential.

Since the fully gapped BCS solution enforces equal numbers of each species, it was incorrectly suggested [2] that one might stabilize the gapless phase by enforcing constraints on the particle numbers. Indeed, by enforcing unequal numbers of each species, one forbids the formation of a fully gapped BCS state, but admits "breached pair" states in which the excess in one species can be accommodated by the breach. In the QCD context, a similar argument has been made by imposing charge neutrality [3, 4]. Recently, however, Bedaque, Caldas and Rupak [8, 9] pointed out that a spatially mixed phase may be energetically preferable: this rules out the first possibility [2] but may not affect the QCD case due to the long-range gauge interactions.

Here we clarify, broaden, and correct this discussion. We conclude that:

- For extensive systems, one can not stabilize a state by imposing different global constraints (such as fixed particle number). For such systems, the composition of the state can be completely determined from an analysis of the grand canonical ensemble. The specific examples considered in [2] are accordingly unstable.
- There are, however, closely related examples of extensive systems that exhibit breached pair superfluidity. We exhibit some below. These states are thermodynamically stable at fixed chemical potentials.

Our considerations do not apply directly to non-extensive systems. Charge conservation or color neutrality constraints enforced by long-range gauge forces might stabilize BP phases. (Of course, the possibility of a competing mixed phase must still be considered quantitatively.)

THERMODYNAMIC STABILITY

In the context of two component fermionic systems as considered in [2, 8, 9], three competing homogeneous phases have been considered: a normal state of free fermions (N), a fully gapped superfluid phase (BCS), and a gapless BP phase. The BCS phase has complete pairing between the two species, and thus enforces equal densities. The other phases admit differing densities.

Upon solving the self-consistency conditions (gap equations), one commonly finds that over a range of chemical potentials there are three distinct solutions. To determine the thermodynamic stability in this grand canonical ensemble, one must minimize the grand thermodynamic potential or, equivalently, maximize the pressure of the system. Typically, two of the three solutions are minima on either side of the third BP state which is a local maximum: Fig. 5 shows a typical potential. The gapless states found in [2, 6] correspond to local maxima, thus the competing state with larger gap parameter Δ has higher pressure and renders the BP state unstable in the grand canonical ensemble.

In these previous models, the stable solution with larger Δ was always a fully gapped BCS state. In the

models we present, this stable state is a gapless BP state.

If the stable solution is fully gapped, however, then it has equal densities and one may forbid this BCS state by fixing unequal densities in the canonical ensemble. Furthermore, upon comparing the Helmholtz free energies H—which must be minimized in this ensemble—one may find that the "unstable" BP state is favored over the normal state N.

This apparent contradiction in the stability analysis based on different ensembles can be resolved by considering a mixed phase [8, 9] which has an even lower Helmholtz free energies H. That such a resolution is always possible, however, may not be apparent; indeed, it is generally hard to determine the mixed phase explicitly. By using general properties [10] of the grand thermodynamic potential Ω, however, one can argue that such a solution is always possible, as follows. By definition,

$$-PV = \Omega(\vec{\mu}) = \min(H - \vec{\mu}\cdot\vec{N}), \tag{1}$$

where the minimization is over all competing phases. Thus, Ω is a concave function of the chemical potentials $\vec{\mu} = (\mu_a, \mu_b)$. (We consider here fixed $T = 0$, but concavity in T also follows from the maximum entropy principle.) Furthermore, there is a one-to-one correspondence between tangents to this surface and states of fixed particle number

$$\vec{N} = -\frac{\partial\Omega}{\partial\vec{\mu}}. \tag{2}$$

When Ω is not differentiable, there is a cone of possible tangent hyperplanes which contact Ω and which bound Ω from above (see Fig. 1). This cone of tangents describes various possible mixed phases composed of the pure phases (where Ω is differentiable) that intersect at the singularity. To find the state that minimizes H for some fixed constraint $\vec{N} = \vec{N}_0$ one simply forms the hyperplane with gradient $\vec{N}_0$ and drops this until it contacts the surface Ω. The first point of contact will define either a pure or mixed state which satisfies the appropriate constraints. Note, however, that this state also lies on Ω, and hence minimizes Ω for the fixed chemical potentials defined by the contact point. No matter what constraints we apply, there is always a stable state in the grand canonical ensemble that satisfies the constraints.

Note, however, that this argument is valid only for extensive thermodynamic systems. Long-range interactions can render the energy of some pure phases non-extensive (due, for example, to the rapidly diverging Coulomb energy per unit volume V as $V \to \infty$). In such cases, a mixed phase would contain bubbles of limited size. The surface energy of these phase boundaries becomes a volume effect and must therefore be taken into account, even in the thermodynamic limit. This complicates the relation between $\vec{N}$ and $\vec{\mu}$. In the remainder of this letter, we shall consider only finite-range interactions.

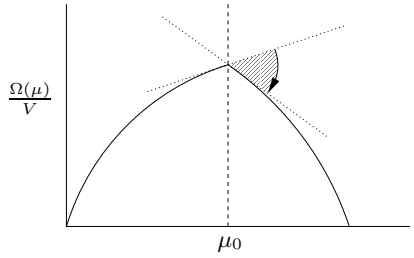

FIG. 1: The cone of tangent (hyper)planes to a thermodynamic potential density $-P = \Omega(\mu)/V$. Immediately to the left of μ_0 is a pure phase with density n_L while immediately to the right is another pure phase with density n_R. The densities are the negative slopes of the tangents at μ_0 according to (2). At $\mu = \mu_0$ there is a continuum of mixed phases: These consist of a volume fraction x at density n_L and the remaining fraction $1 - x$ at density n_R. The average density over all space, $n = xn_L + (1-x)n_R$, lies within $n \in (n_L, n_R)$.

STABLE BREACHED PAIR SUPERFLUIDS

We now demonstrate, by example, how to realize pure BP superfluid states in extensive systems. We shall consider the mean-field analysis of two models, each with two species of fermions a, and b of differing masses $m_a < m_b$:

$$\mathcal{H} = \int \text{đ}^3\vec{p}\left(\frac{p^2}{2m_a}\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{p^2}{2m_b}\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}\right) + \mathcal{H}_I. \tag{3}$$

(By convention, $\text{đ}p \equiv \text{d}p/(2\pi)$, $\vec{p}$ is a vector with magnitude p, and $\hat{a}$ is an operator.) We shall consider these systems in the grand canonical ensemble at zero temperature by minimizing the thermodynamic potential density $\Omega(\mu_a, \mu_b)/V$. It will be natural, however, to use the parameters $p_i^F = \sqrt{2m_i\mu_i}$ in place of the chemical potentials μ_i.

The first model posits a spherically symmetric static two-body potential interaction $V(r)$ between the two species a and b:

$$\mathcal{H}_I = \int \text{d}^3\vec{x}\,\text{d}^3\vec{y}\;\; V\big(|\vec{x}-\vec{y}|\big)\;\; \hat{a}^\dagger_{\vec{x}}\,\hat{b}^\dagger_{\vec{y}}\,\hat{b}_{\vec{y}}\,\hat{a}_{\vec{x}}. \tag{4}$$

Defining $m_\pm = 2m_am_b/(m_b \pm m_a)$ and $\mu_\pm = (\mu_a \pm \mu_b)/2$,

$$\epsilon_p^\pm \equiv \tfrac{1}{2}\left[\frac{p^2}{2m_a} - \mu_a\right] \pm \tfrac{1}{2}\left[\frac{p^2}{2m_b} - \mu_b\right] = \frac{p^2}{2m_\pm} - \mu_\pm, \tag{5}$$

and considering only homogeneous (translationally invariant) and isotropic phases, we find that extrema of (1) satisfy the gap equation

$$\Delta_p = -\int_R \text{đ}^3\vec{q}\;\; \tilde{V}\big(|\vec{p}-\vec{q}|\big)\frac{\Delta_q}{2\sqrt{(\epsilon_q^+)^2 + \Delta_q^2}}, \tag{6}$$

where $\tilde{V}(p)$ is the Fourier transform of $V(r)$. The integral (6) runs over the region R outside any "breach".

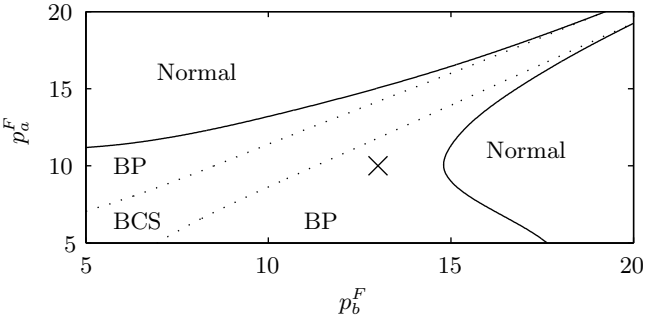

FIG. 2: Qualitative $T = 0$ phase diagram for static two-body potential scattering model with a Gaussian potential $V(r) \propto \exp(-r^2/2\lambda^2)$. All momentum scales are in units of $\hbar/\lambda$ and all energy scales are in units of $\hbar^2/(m_+\lambda^2)$. The mass ratio is $m_b/m_a = 50$ and the coupling strength has been chosen so that $2m_+\Delta_{p_0}/p_0^2 = 0.1$ at the point marked "×" where $(p_b^F, p_a^F) = (13, 10)$ to ensure weak-coupling. (This ratio is less that 1 at all points in this diagram.) Note that the lower BP region has more heavy particles b while the upper BP region has fewer heavy particles. The upper type may be realized in the QCD context [4].

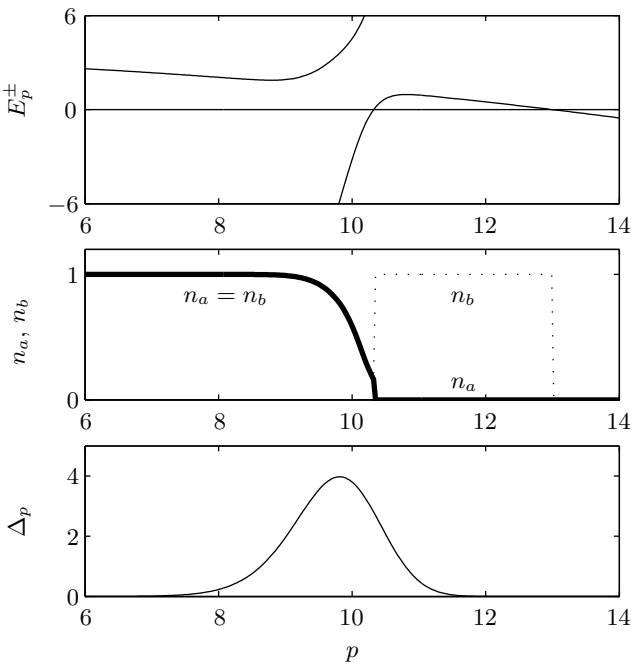

FIG. 3: Quasi-particle dispersions $E_p^\pm$ (top), occupation numbers n_a and n_b (middle), and gap parameter Δ_p for a sample BP state at $(p_b^F, p_a^F) = (13, 10)$. All units and parameters are described in Fig. 2. Notice that there are two "Fermi" surfaces at $p \approx 10.3$ and $p = 13$. The first occurs where Δ_p becomes too small to support a gap, while the second is simply the Fermi surface for b (which is unaffected by the pairing). The "breach" occurs between these surfaces and only the region R outside contributes to the gap equation (6).

R contains momenta where the two quasiparticle dispersions $E_p^\pm$

$$E_p^\pm = \epsilon_p^- \pm \sqrt{(\epsilon_p^+)^2 + \Delta_p^2}, \tag{7}$$

have opposite sign. (See [2] for further details about the generic breach structure.) Note from (6) that Δ_p is generally largest about p_0, where $\epsilon_{p_0}^+ = 0$.

Equation (6) can be solved numerically to find extremal points of the thermodynamic potential. Over this set of self-consistent solutions, one can minimize Ω to determine the phase structure.

We have done this for a variety of interactions, and find similar qualitative structure: a central strip of fully gapped BCS-like phase about $p_a^F = p_b^F$, with normal unpaired phases outside (see Fig. 2.) Depending on the model parameters, these phases may be separated by a region of BP superfluid phase. To verify that these indeed contain gapless modes we plot in Fig. 3 a sample set of occupation numbers, quasiparticle dispersions, and the gap parameter Δ_p,

$$\Delta_p = \int \mathrm{d}^3\vec{q} V(|\vec{p} - \vec{q}|)\langle \hat{b}_{\vec{p}}\hat{a}_{-\vec{p}}\rangle. \tag{8}$$

The presence of gapless fermion modes depends crucially on the momentum structure of Δ_p. In particular, there must be at least two distinct regions in momentum space: one with Δ_p large enough to support the superfluid, and another with Δ_p small enough that pairing does not appreciably affect the normal free-fermion behaviour. Having such a structure, however, does not guarantee the stability of the phase: the phase must also have higher pressure than the normal phase.

Consider a state with $p_a^F = p_b^F = p_0$: This is in the standard BCS phase and, as is well known, always admits a stable gapped solution. Now consider adjusting the chemical potentials so as to increase the Fermi surface p_b^F. This stresses the system and lowers the pressure relative to the normal phase. Eventually, either before or after a transition to a BP state, the pressure becomes negative and there is a first order phase transition to the normal phase of unpaired fermions.

One can consider the point just before this transition occurs: if $\Delta_{p_b^F}$ is sufficiently large, the state will still be gapped (BCS) and no BP state will occur. On the other hand, if $\Delta_{p_b^F}$ is small, then it will not appreciably affect the dispersions and one will find a gapless Fermi surface coexisting with the superfluid phase and a stable BP state. As was emphasized in [1], the cost associated with shifting the Fermi sea p_b^F can be reduced by increasing the mass m_b. Thus, as long as Δ_p falls off sufficiently quickly, one can choose m_b so that the transition will occur with $\Delta_{p_b^F}$ small enough to support the BP phase. The states shown in Fig. 2 and Fig. 3 have a mass ratio $m_b/m_a = 50$.

We have examined other forms of interaction with longer range and find similar results, though the ex-

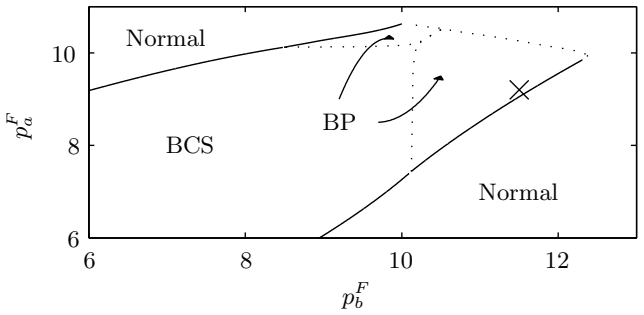

FIG. 4: $T = 0$ phase diagram for the model (9). We have used a hard cutoff $f(p) \sim \theta(p - 10\sigma)$ that has been smoothed over the range from $p \in (9.7, 10.3)$. All momenta are expressed in units of σ where 10σ is the cutoff scale, and all energies are expressed in units of $\sigma^2/(2m_+)$. The mass ratio is $m_b/m_a = 4$ and the coupling g has been chosen so that $2m_+\Delta/p_0^2 = 0.2$ at $p_a^F = p_b^F = p_0 = 10\sigma$ to ensure weak-coupling. (This ratio is less that 1 at all points in this diagram). The solid lines correspond to first order phase transitions—the mixed phases of [8, 9] would be found on these lines—and the dotted lines correspond to smooth, higher order transitions. Close to the phase transition fluctuations beyond mean-field may play an important role. We have also found cases in model (4) where the BCS-BP transition appears to be weakly first order. These issues will be discussed more fully elsewhere [11]. The sample state in Fig. 5 at $(p_b^F, p_a^F) = (11.5, 9.2)$ is marked "×".

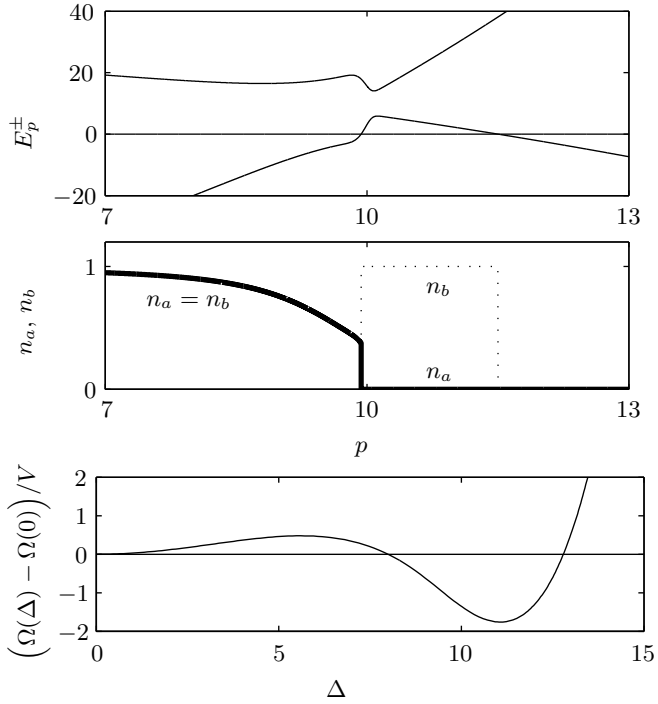

FIG. 5: Quasi-particle dispersions $E_p^\pm$ (top), and occupation numbers n_a and n_b (middle), in a sample BP state. This state has gap parameter $\Delta \approx 11$ which is the global minimum of the grand thermodynamic potential density $\Omega(\Delta)/V$ (bottom) as defined in (10). The maximum at $\Delta \approx 5.6$ corresponds to an unstable BP state. These figures correspond to the point $(p_b^F, p_a^F) = (11.5, 9.2)$ in Fig. 4.

act structure of Δ_p varies. For example, longer-range forces (such as a screened Coulomb interaction) exhibit a plateau for low momenta, and fall more slowly to the right of p_0, requiring larger mass ratios. In principle, however, a BP state can always be realized in these models with a sufficiently large mass ratio.

Since the variational states of model (4) are parameterized by a variable *function* Δ_p, the set of states over which the minimization (1) must consider is enormous, and we cannot be certain to have found the global minimum. We have searched for stable fixed-points of the gap equation (6) and compared them, so our results for this model are consistent and plausible, but not rigorous.

To construct a model for which we can be certain of the phase structure, we fall back to the type of factorized, cutoff interaction often considered in BCS models:

$$\mathcal{H}_I = g \int \text{đ}^3\vec{p}\, \text{đ}^3\vec{q}\ \ f(p)f(q)\ \ \hat{a}^\dagger_{\vec{p}}\, \hat{b}^\dagger_{-\vec{p}}\, \hat{b}_{-\vec{q}}\, \hat{a}_{\vec{q}}. \qquad (9)$$

We can choose the cutoff function $f(p)$ to mimic the behaviour of Δ_p in the more physical model (4).

The advantage of this model is that the variational states are parameterized by a single number Δ which is the expectation value of the operator $\hat{\Delta} = g \int \text{đ}^3\vec{p}\ \ f(p)\hat{b}_p\hat{a}_{-p}$. (The momentum dependence has been replaced by the function $f(p)$.) Thus, one can find the *global* minimum by plotting (see Fig. 5) the thermodynamic potential density

$$\frac{\Omega(\Delta)}{V} = \min_{\langle\theta|\hat{\Delta}|\theta\rangle=\Delta} \langle\theta|\hat{\mathcal{H}} - \mu_a\hat{n}_a - \mu_b\hat{n}_b|\theta\rangle, \qquad (10)$$

where the minimization is over all BCS style ansatz $|\theta\rangle$ (equivalent to the mean-field approximation) with given expectation Δ. This minimization is equivalent to comparing *all* solutions to the gap equation

$$\Delta = -\frac{g}{2}\int_R \text{đ}^3\vec{q}\ \ \frac{\Delta f(q)}{\sqrt{\epsilon_+^2(q) + \Delta^2}}. \qquad (11)$$

From this we conclude that, within the space of homogeneous phases at zero temperature in the mean-field approximation, this model has the phase diagram shown in Fig. 4. We plot the properties of a sample BP state in Fig. 5 to illustrate that there are indeed gapless modes.

To model Δ_p more accurately one might use a function $f(p)$ where the location of the cutoff stays near p_0. This introduces an inconsistency in the thermodynamics because $f(p)$ is really a property of the Hamiltonian, while p_0 depends on the chemical potentials μ_i, thus $N \neq -\partial\Omega/\partial\mu$. For small coupling and high densities, these spurious dependencies become small and the resulting phase diagram is qualitatively like Fig. 2.

Physical realizations of a stable BP phase require either non-extensivity, or a momentum dependent interaction with a large mass ratio. The former case may occur in high-density QCD [3, 4] where gauge interactions stabilize the state. The latter case may occur in a quantum gas of cold neutral atoms operating near Feshbach resonance with effective masses tuned by a laser lattice [5], or in a system of trapped ions with dipolar interactions [12].

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